

Complete Systems of Concomitants of the Three-Point
and the Four-Point in Elementary Geometry

DISSERTATION

Submitted to the Board of University Studies of The Johns Hopkins University in
Conformity with the Requirements for the Degree of Doctor of Philosophy

BY

CHARLES HENRY RAWLINS, JR.

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INTRODUCTION.

In this discussion, two point-sets, containing three and four points respectively, are subjected to three transformations of elementary geometry; and complete systems of invariants and covariants, corresponding to the respective transformations, are derived. In the process we obtain some interesting geometric applications of the theory of binary forms and of symmetric functions.

PART I.

INVARIANTS UNDER TRANSLATION.

Section (a): The Three-Point.

Consider first the three-point under translation. Let the points have complex coordinates α, β, γ , respectively, the roots of a cubic

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0.$$

Since translation is effected by adding to each root the same vector p , the invariants of the three-point under translation are the geometric equivalents of the invariants of the cubic under the transformation

$$x' = x + p.$$

Invariants under such a transformation are known, in the theory of binary forms, as *seminvariants*. Their complete system for the cubic is known* to consist of the coefficients of the cubic when so transformed that its second term vanishes, and the discriminant

$$\Delta \equiv (a_0a_3 - a_1a_2)^2 - 4(a_0a_2 - a_1^2)(a_1a_3 - a_2^2).$$

Substituting $x - (a_1/a_0)$ for x in the cubic, we obtain

$$A_0x^3 + 3A_2x + A_3 = 0,$$

* Elliott, "Algebra of Quantics," pp. 140, 162.

the form desired, where, except for factor a_0 ,

$$A_0 = a_0, \quad A_2 = a_0 a_2 - a_1^2, \quad A_3 = a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3.$$

Connecting these is the syzygy

$$A_0^2 \Delta = 4A_2^3 + A_3^4$$

However, without loss of generality, we can consider A_0 (or a_0) as having the fixed value unity, in which case Δ is expressed *integrally* in terms of A_2 and A_3 . Therefore

The complete system of invariants of the three-point under translation consists of A_2 and A_3 .

Since the *shape* of the three-point is an invariant property under translation, the vanishing of A_2 or A_3 indicates some condition on the shape.

Vanishing of A_2 :—If $A_2 = 0$, the cubic is

$$x^3 + A_3 = 0,$$

with roots in ratio $1:\omega:\omega^2$, where ω is one of the complex cube roots of unity. The points form, therefore, the vertices of an equilateral triangle.

Vanishing of A_3 :—If $A_3 = 0$, the cubic is

$$x^3 + 3A_2 x = 0$$

with roots in ratio $0:1:-1$. Hence the points are collinear, with one midway between the others.

A_1 being absent, the sum of the roots is zero and therefore the origin is at the centroid.

Section (b): The Four-Point.

Similarly, *the complete system of seminvariants of the four-point*

$$a_0 x^4 + 4a_1 x^3 + 6a_2 x^2 + 4a_3 x + a_4 = 0,$$

consists of *

$$A_0 = a_0 (=1), \quad A_2 = a_0 a_2 - a_1^2, \quad A_3 = a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3,$$

$$I = a_0 a_4 - 4a_1 a_3 + 3a_2^2, \quad J = a_0 a_2 a_4 + 2a_1 a_2 a_3 - a_0 a_3^2 - a_1^2 a_4 - a_2^3.$$

However, since

$$A_4 = a_0^3 a_4 - 4a_0^2 a_1 a_3 + 6a_0 a_1^2 a_2 - 3a_1^4 = a_0^2 I - 3A_2^2,$$

we will use it instead of I .

Again the origin is at the centroid, and the vanishing of an invariant is a condition on the *shape* of the configuration.

Vanishing of A_2 :—Let $\alpha, \beta, \gamma, \delta$ be the roots of

$$A_0 x^4 + 6A_2 x^2 + 4A_3 x + A_4 = 0.$$

* Elliott, *supra*, p. 170.

Then, if $A_2=0$, $\Sigma\alpha\beta=0$; so that, since $\Sigma\alpha=0$, then $\Sigma\alpha^2=0$. Refer the system to rectangular axes through the origin, so that $\alpha=X_1+iY_1$, $\beta=X_2+iY_2$, etc. Then

$$\sum_{j=1}^4 (X_j + iY_j)^2 = 0.$$

Expanding and equating real parts,

$$\Sigma X^2 - \Sigma Y^2 = 0.$$

But

$$\Sigma X^2 + \Sigma Y^2 = \Sigma R^2,$$

where the R 's are distances from the origin to the respective points, constants in the present discussion. Therefore

$$\Sigma X^2 = 1/2 \Sigma R^2 = \text{constant.} \quad (1)$$

If the points are of unit mass, ΣX is the moment of inertia of the system about the Y -axis. But, the direction of this axis having been arbitrarily chosen, (1) tells us that the moment of inertia is the same for all axes through the centroid, thus making the "ellipse of inertia" * a circle.

Vanishing of A_3 :—If $A_3=0$, the quartic is

$$x^4 + 6A_2x^2 + A_4 = 0,$$

with factors $x^2 - m_1$, $x^2 - m_2$, and roots $\pm\sqrt{m_1}$, $\pm\sqrt{m_2}$. Hence the points form the vertices of a parallelogram.

Vanishing of A_4 :—If $A_4=0$, one root of the quartic is zero. Hence one point lies at the centroid of the other three.

Vanishing of J :—The vanishing of J is known to be the condition that the four roots form harmonic pairs; or, geometrically, that the four points lie on one of a set of three mutually orthogonal circles, at its intersections with the other two.

Since *infinity* is unaltered by a finite translation, it has not been necessary, in the foregoing, to consider it as part of the apparatus employed.

PART II.

MONOGENIC CONCOMITANTS IN THE COMPLEX PLANE.

Section (a): General Theory.

We will consider next the monogenic concomitants of the point-sets in the complex plane, that is, concomitants not involving the conjugates of any of

* Routh, "Rigid Dynamics," Chap. I.

the complex quantities used. Then the most general (linear) transformation possible is

$$x' = (ax + b) / (cx + d)$$

the product of an even number of inversions. Since infinity is not, in general, a fixed point of this transformation, it must be considered explicitly, that is:

We must discuss not merely the concomitants of a point-set, *but the concomitants of the point at infinity and the point-set*. Hence we use the theory of a binary cubic (quartic) and a linear form. From this theory we learn * that *the complete system of the cubic (quartic) and the linear form consists of the linear form, the complete system of the cubic (quartic) and the polars of this system with respect to the linear form*.

The linear form being the equation of infinity, its polar operator, under the present notation, is simply d/dx .

Section (b): *The Three-Point.*

The complete system of the cubic consists of †

the cubic itself:

$$C = a_0x^3 + 3a_1x^2 + 3a_2x + a_3,$$

the Jacobian:

$$G = A_3x^3 + \dots,$$

the Hessian:

$$H = A_2x^2 + \dots,$$

the discriminant: $\Delta = (a_0a_3 - a_1a_2)^2 - 4(a_0a_2 - a_1^2)(a_1a_3 - a_2^2)$.

Successive derivatives of these are (neglecting constant factors):

$$\begin{array}{lll} C' = a_0x^2 + 2a_1x + a_2, & C'' = a_0x + a_1, & C''' = a_0 \text{ (negligible)}, \\ G' = A_3x^2 + \dots, & G'' = A_3x + \dots, & G''' = A_3, \\ H' = A_2x + \dots, & H'' = A_2. & \end{array}$$

As in Part I, Section (a), Δ is expressible integrally in terms of A_2 and A_3 .

Complete System.—Thus the complete system contains

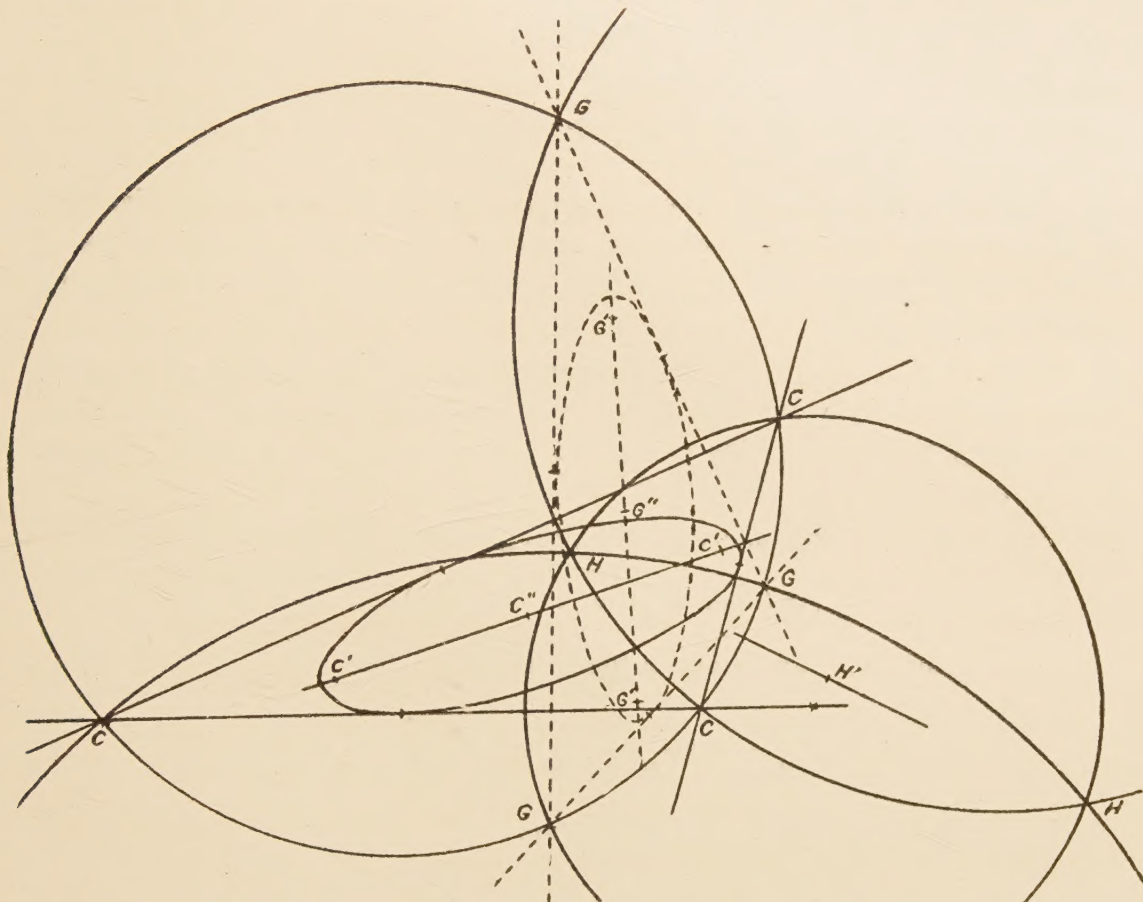
two cubics:	$C, G,$
three quadratics:	$C', G', H,$
three linear forms:	$C'', G'', H',$
two invariants:	$G''' = A_3, H'' = A_2.$

Geometric Equivalents.—We will state, without demonstration, the geometric equivalents (see figure).

* Grace & Young, "Algebra of Invariants," 138 A.

† Salmon, "Higher Algebra"; Elliott and Grace & Young, *supra*.

The G -points are cut out of the circle on the C -points by means of the three Apollonian circles, that is, circles each on one C -point and drawn about the other two. The Apollonian circles are members of a pencil whose fixed points are the H -points. C' are the foci of the ellipse inscribed in the C -triangle with its center at the centroid, and C'' is the centroid. G' and G'' are similarly related to the G -triangle. H' is midway between the H -points. If $A_2=0$, an H -point is at infinity. If $A_3=0$, a G -point is at infinity.



Section (c): The Four-Point.

The complete system of the quartic consists of
the quartic itself: $Q = a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4$,

the Jacobian: $G = A_3x^6 + \dots$,

the Hessian: $H = A_2x^4 + \dots$,

two invariants:

$$I = a_0a_4 - 4a_1a_3 + 3a_2^2, \quad J = a_0a_2a_4 + 2a_1a_2a_3 - a_0a_3^2 - a_1^2a_4 - a_2^3.$$

Complete System.—Taking derivatives, the complete system of the four-point is

one sextic:	$G,$
one quintic:	$G',$
three quartics:	$Q, G'', H,$
three cubics:	$Q', G''', H',$
three quadratics:	$Q'', G^{(4)}, H'',$
three linear forms:	$Q''', G^{(5)}, H''',$
four invariants:	$G^{(6)}=A_3, H^{(4)}=A_2, I, J.$

Geometric Equivalents.—We will define a few of the geometric equivalents of these forms. G is the double points of the involutions formed by taking the Q -points in pairs. The derived forms can be interpreted by the following theorem: *

Given $\phi(p_1, p_2, \dots, p_r)=0$, homogeneous in the p 's, as the equation of a curve, where the p 's are the distances from given points $a_1, a_2, \dots, a_r$, respectively, to a line of the curve, the foci of the curve are the roots of $\phi(x-a_1, x-a_2, \dots, x-a_r)=0$.

Suppose now n points, $a_1, a_2, \dots, a_n$, given by

$$f=(x-a_1)(x-a_2)\dots(x-a_n)=0.$$

The derived equation can be put in the form

$$f'=\frac{1}{x-a_1}+\frac{1}{x-a_2}+\dots+\frac{1}{x-a_n}=0.$$

By the theorem, f' gives the foci of the curve

$$\sum_{i=1}^n 1/p_i=0,$$

which is of class $n-1$, is on the $n(n-1)/2$ joins of the n points and touches each join at its mid-point.

Hence G' is the foci of such a curve on the joins of the G -points, and similarly for the other derived forms.

If $A_2=0$, an H -point is at infinity. If $A_3=0$, a G -point is at infinity.

* F. Morley, Lectures 1915-16, Johns Hopkins University.

PART III.

CONCOMITANTS IN METRICAL GEOMETRY.

Coming now to the realm of metrical geometry, we will derive a complete system of curves symmetrically related to the members of a point-set. We assume barycentric coordinates, the line at infinity, and the two imaginary circular points (the absolute) on this line.

Section (a): The Three-Point.

Take the three points as vertices of the reference triangle, and denote by λ_i the square of the length of side lying on $x_i=0$. Then, any curve connected metrically and symmetrically to the points is expressed as a function of two rows,

$$\begin{array}{ccc} x_0, & x_1, & x_2, \\ \lambda_0, & \lambda_1, & \lambda_2, \end{array}$$

which remains unchanged, to within the algebraic sign, when λ_i and x_i are interchanged with λ_j and x_j , respectively.

Thus, our problem reduces to that of deriving a complete system of such functions.

Section (b): The Four-Point.

If the four points are referred to their diagonal triangle as reference triangle, their coordinates are of the form

$$b_0, \pm b_1, \pm b_2,$$

and the entire set is defined by three quantities a_0, a_1, a_2 , such that $a_i=b_i^2$. Then a curve of the kind desired is expressed as a function of three rows,

$$\begin{array}{ccc} x_0, & x_1, & x_2, \\ \lambda_0, & \lambda_1, & \lambda_2, \\ a_0, & a_1, & a_2, \end{array}$$

which remains unchanged, to within the algebraic sign, when x_i, λ_i, a_i , are interchanged with x_j, λ_j, a_j , respectively.

We must derive, then, a complete system of these functions.

Since the three-rowed functions include, as a special case, those of two rows, one algebraic investigation will suffice to determine the complete systems of both classes of functions.

The functions are classified as (1) symmetric, (2) alternating, according as the algebraic sign (1) is not, (2) is, changed when a permutation of elements is made.

Section (c): Symmetric Functions.

The theory of symmetric functions of one row has been thoroughly developed.* The theory for several rows has been investigated by Junker and MacMahon. To some of the articles of the former † reference has been made in the preparation of the following paragraphs:

Symmetric functions of the kind under consideration will be of the form

$$J = \sum \lambda_0^{\alpha} \lambda_1^{\alpha'} \lambda_2^{\alpha''} a_0^{\beta} a_1^{\beta'} a_2^{\beta''} x_0^{\gamma} x_1^{\gamma'} x_2^{\gamma''},$$

the exponents being positive integers or zero. In certain cases (for instance, when $\alpha = \alpha'$, $\beta = \beta'$, $\gamma = \gamma'$) the function has only three distinct terms. It will then be indicated by the symbol $\sum^3$, or by placing its characteristic term in round brackets. Attention is called also to the products $\lambda_0 \lambda_1 \lambda_2$, $a_0 a_1 a_2$, $x_0 x_1 x_2$, which are symmetric functions of *one term only*.

We will prove that J can be expressed integrally in terms of three-term functions.

Let
$$p_i = \lambda_i^{\alpha} a_i^{\beta} x_i^{\gamma}, \quad q_i = \lambda_i^{\alpha'} a_i^{\beta'} x_i^{\gamma'}, \quad r_i = \lambda_i^{\alpha''} a_i^{\beta''} x_i^{\gamma''}.$$

Then
$$\begin{aligned} J &= \sum^3 p_0 (q_1 r_2 + q_2 r_1) \\ &= \sum^3 p_0 \{ \sum^3 q_0 (r_1 + r_2) - [q_0 (r_1 + r_2) + r_0 (q_1 + q_2)] \} \\ &= \sum^3 p_0 \{ \sum^3 q_0 ((r) - r_0) - [q_0 ((r) - r_0) + r_0 ((q) - q_0)] \} \\ &= \sum^3 p_0 [(q)(r) - (qr) - (r)q_0 + 2q_0 r_0 - (q)r_0] \\ &= (p)(q)(r) - (p)(qr) - (pq)(r) + 2(pqr) - (q)(pr). \end{aligned}$$

Therefore

(A) *The complete system will contain no six-term symmetric functions.*

Since J has as a factor the symmetric function $\lambda_0^m \lambda_1^m \lambda_2^m$, where m is the smallest of the α 's (similarly for a and x):

(B) *The complete system will contain no symmetric functions with more than two λ 's, two a 's and two x 's in a term.*

Consider

$$\begin{aligned} (Ax^n) &= A_0 x_0^n + A_1 x_1^n + A_2 x_2^n, \\ (Ax^{n-1}) &= A_0 x_0^{n-1} + A_1 x_1^{n-1} + \dots, \\ (Ax^{n-2}) &= A_0 x_0^{n-2} + \dots, \\ (Ax^{n-3}) &= A_0 x_0^{n-3} + \dots, \end{aligned}$$

* Salmon, "Higher Algebra."

† *Math. Ann.*, Vols. XLIII, XLV.

where the A 's are expressions in λ , a , and x , consistent with the three-term form in which used.

Eliminating the A 's, we obtain

$$\begin{vmatrix} (Ax^n) & x_0^n & x_1^n & x_2^n \\ (Ax^{n-1}) & x_0^{n-1} & x_1^{n-1} & x_2^{n-1} \\ (Ax^{n-2}) & x_0^{n-2} & x_1^{n-2} & x_2^{n-2} \\ (Ax^{n-3}) & x_0^{n-3} & x_1^{n-3} & x_2^{n-3} \end{vmatrix} = 0,$$

which, after the removal of certain factors, gives

$$(Ax^n) = (x)(Ax^{n-1}) - (x_1x_2)(Ax^{n-2}) + x_0x_1x_2(Ax^{n-3}).$$

Repeating the process on (Ax^{n-1}) , etc., we obtain finally (Ax^n) in terms of (Ax^2) , (Ax) , (A) , (x) , (x_1x_2) and $x_0x_1x_2$. Since the process applies equally well to the λ 's and the a 's.

(C) *The complete system will contain no symmetric functions in which an element occurs to a higher degree than the second.*

We state for reference the following special case of the above formula:

$$(Ax^3) = (x)(Ax^2) - (x_1x_2)(Ax) + x_0x_1x_2(A). \quad (1)$$

Since by (B) no term is to contain more than two elements of a kind, the total degree in x (λ or a) must not exceed four. But, if the function be $(A_0x_1^2x_2^2)$, of total degree 4 in x , we can use the important identities

$$x_ix_j = (x_1x_2) - (x)x_k + x_k^2 \quad (i, j, k=0, 1, 2; i \neq j \neq k), \quad (2)$$

and obtain

$$\begin{aligned} (A_0x_1^2x_2^2) &= (A)(x_1x_2)^2 + (x)^2(Ax^2) + (Ax^4) - 2(x)(x_1x_2)(Ax) \\ &\quad - 2(x_1x_2)(Ax^2) - 2(x)(Ax^3), \end{aligned}$$

in which (Ax^4) and (Ax^3) are reducible by (C).

A function of total degree 3 in x is of the type $\Sigma A_0x_1^2x_2$. This necessarily contains six terms and is therefore excluded by (A). Hence:

(D) *The complete system will contain no symmetric functions in which the total degree in λ , a , or x is greater than 2, except the special functions $\lambda_0\lambda_1\lambda_2$, $a_0a_1a_2$, $x_0x_1x_2$.*

For reference, we restate (2) in different form:

$$x_i^2 = x_jx_k + (x)x_i - (x_1x_2) \quad (i, j, k=0, 1, 2; i \neq j \neq k). \quad (3)$$

A function can contain x to total degree 2 in two ways:

$$(Ax^2) \text{ or } (A_0x_1x_2).$$

But, by (2),

$$(A_0x_1x_2) = (A)(x_1x_2) - (x)(Ax) + (Ax^2).$$

Hence:

(E) *As members of the complete system of symmetric functions, the forms $(A_0x_1x_2)$ and (Ax^2) (similarly for the λ 's and a 's) are mutually exclusive.*

We will adopt the form $(A_0x_1x_2)$.

Denoting by s_{pqr} a three-term symmetric function of total degree p in λ , q in a , and r in x ; principles (A) to (E), inclusive, leave for individual consideration

s_{001}	s_{010}	s_{100}
s_{002}	s_{020}	s_{200}
s_{003}	s_{030}	s_{300}
s_{011}	s_{101}	s_{110}
s_{012}	s_{102}	s_{120}
s_{021}	s_{201}	s_{210}
s_{022}	s_{202}	s_{220}
s_{111}		
s_{112}	s_{121}	s_{211}
s_{122}	s_{212}	s_{221}
s_{222}		

We need consider only the first column in detail, because the others are obtained from it by interchange of letters.

$s_{001} = (x)$ and $s_{002} = (x_1x_2)$ are obviously irreducible.

$s_{003} = x_0x_1x_2$, $s_{011} = (ax)$, $s_{012} = (a_0x_1x_2)$, and $s_{021} = (a_1a_2x_0)$ are found to be irreducible by a test which will be illustrated by use on $s_{022} = (a_1a_2x_1x_2)$.

This, if reducible, will be a sum of products of irreducible functions, each product of degree 022. Assume then

$$(a_1a_2x_1x_2) = k(a)(a_0x_1x_2) + l(x)(a_1a_2x_0) + m(a_1a_2)(x_1x_2) + n(ax)^2 \\ + p(a)^2(x_1x_2) + q(x)^2(a_1a_2) + r(a)(x)(ax) + s(a)^2(x)^2.$$

Then, by substituting sets of numerical values for the a 's, and x 's we obtain a sufficient number of equations, simultaneous in k, l, m , etc., to solve for their values. The operation is greatly simplified by a selection of numbers which causes several terms of the assumed identity to vanish.

For instance, substituting

$$1, -1, 0; 1, 1, -2 \text{ for } a_0, a_1, a_2; x_0, x_1, x_2,$$

respectively; (a) , (ax) , and (x) vanish, and we obtain

$$-1 = 3m, \text{ whence } m = -1/3.$$

Similarly, from

$$\begin{aligned} 1, -1, 0; 1, -1, 0; \quad n=1/3, \\ 1, 0, 0; 1, -1, 0; \quad p=1/3, \text{ etc.} \end{aligned}$$

As a result

$$\begin{aligned} \text{(F)} \quad s_{022} = (a_1 a_2 x_1 x_2) = & -1/3(a)(a_0 x_1 x_2) - 1/3(x)(a_1 a_2 x_0) \\ & - 1/3(a_1 a_2)(x_1 x_2) + 1/3(ax)^2 \\ & + 1/3(a)^2(x_1 x_2) + 1/3(x)^2(a_1 a_2) - 1/3(a)(x)(ax), \end{aligned}$$

the correctness of which has been tested by various numerical substitutions.

If such substitutions fail to satisfy the expression, or if contradictions arise in solving for the coefficients, the assumption of an identity is false and the function is irreducible. As an illustration, assume

$$s_{012} = (a_0 x_1 x_2) = k(x)(ax) + l(a)(x_1 x_2) + m(a)(x)^2.$$

From $1, -1, 0; 1, 0, 0; \quad k=0,$

and from $1, 1, 0; 1, -1, 0; \quad l=0,$

whence $(a_0 x_1 x_2) = m(a)(x)^2,$

which is obviously untrue, whatever the value of m .

In this way s_{003} , s_{011} , s_{021} and $s_{111} = (\lambda ax)$ are proved irreducible.

Similar methods show that

$$\begin{aligned} \text{(G)} \quad s_{112} = (\lambda_0 a_0 x_1 x_2) = & 1/3(\lambda)(a_0 x_1 x_2) + 1/3(a)(\lambda_0 x_1 x_2) \\ & - 1/3(x)(\lambda ax) + 2/3(\lambda a)(x_1 x_2) \\ & + 1/3(\lambda x)(ax) - 1/3(\lambda)(a)(x_1 x_2), \end{aligned}$$

which stands the test for an identity.

(H) $s_{122} = (\lambda_0 a_1 a_2 x_1 x_2)$ reduces by substituting $a_1 a_2$ for a_0 , $a_2 a_0$ for a_1 , $a_0 a_1$ for a_2 in (G). Similarly,

(K) $s_{222} = (\lambda_1 \lambda_2 a_1 a_2 x_1 x_2)$ reduces by substituting $\lambda_1 \lambda_2$ for λ_0 , etc., in (H).

Complete System.—We have, therefore, the following complete system of symmetric functions:

Of λ alone	s_{100}	s_{200}	s_{300}
Of a alone	s_{010}	s_{020}	s_{030}
Of x alone	s_{001}	s_{002}	s_{003}
Of a and x	s_{011}	s_{012}	s_{021}
Of λ and x	s_{101}	s_{102}	s_{201}
Of λ and a	s_{110}	s_{120}	s_{210}
Of λ , a and x	s_{111}		

Furthermore, the foregoing processes, (A) to (K) inclusive, enable us to express any given symmetric function in terms of the members of the complete system.

Section (d): Alternating Functions.

Using the same notation as in Section (c), the most general alternating function of three rows is

$$K = p_0q_1r_2 + p_1q_2r_0 + p_2q_0r_1 - p_0q_2r_1 - p_1q_0r_2 - p_2q_1r_0.$$

This is the same as

$$|pqr| \equiv \begin{vmatrix} p_0 & q_0 & r_0 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix}.$$

Taking K however in the form $\sum^3 A_p p$, where $A_0 = q_1r_2 - q_2r_1$, etc., it is easily proved that

(B') *The complete system will contain no alternating functions with more than two λ 's, two a 's and two x 's in a term.*

(C') *The complete system will contain no alternating functions in which an element occurs to a degree higher than the second.*

Functions containing λ , a or x to total degree 4 can be reduced by (2), Section (c), to functions of total degree 2 in that quantity. Functions of total degree 3 in λ , a or x are not reducible by this means. Hence

(D') *The complete system will contain no alternating functions in which the total degree in λ , a or x is greater than 3.*

(E) can be restated:

(E') *As members of the complete system, $\sum^3 A_i x_1 x_2$ and $\sum^3 A_i x_0^2$ (similarly for the λ 's and a 's) are mutually exclusive.*

A further important principle is obtained by considering the determinant

$$\begin{vmatrix} p_0 & q_0 & r_0 & 1/3 \\ p_1 & q_1 & r_1 & 1/3 \\ p_2 & q_2 & r_2 & 1/3 \\ (p) & (q) & (r) & 1 \end{vmatrix}.$$

This vanishes identically because the last row is the sum of the others. Hence, expanding:

$$|pqr| = 1/3(p)|q r 1| - 1/3(q)|p r 1| + 1/3(r)|p q 1|.$$

This reduces any alternating function except those in which p , q or r equals unity. Therefore:

(L) *The complete system will contain no alternating functions except those whose determinants have a column (or row) of 1's.*

Such functions, of total degree 3 in x , are of the type

$$|ux^2 vx 1|$$

where u and v are products of powers of λ and a . By (2), Section (c),

$$|ux^2 vx 1| = (x_1 x_2) |ux v 1| - (x) |ux v x| + |ux v x^2|.$$

Only the last function in this expansion contains x to total degree as high as 3. Applying (L) to this function,

$$|ux v x^2| = 1/3(ux) |v x^2 1| - 1/3(v) |ux x^2 1| + 1/3(x^2) |ux v 1|.$$

Likewise, in this expansion, only $|ux x^2 1|$ is of total degree 3 in x . Applying (2) to this,

$$|ux x^2 1| = (x_1 x_2) |u x 1| + |u x x^2|.$$

Again, applying (L) to $|u x x^2|$,

$$|u x x^2| = (u) |1 x x^2| - (x) |u x^2 1| + (x^2) |u x 1|.$$

Finally, $|1 x x^2|$ is unaltered by (2), (3) or (L).

If u , v , or both equal unity, the process is merely shortened; the conclusion is the same, namely:

(D') *The complete system will contain no alternating functions in which the total degree in λ , a or x is greater than 2, except possibly $|x^2 x 1|$, $|\lambda^2 \lambda 1|$ and $|a^2 a 1|$.*

Consider the cases where at least two elements occur, each to total degree 2. If w and w' are powers of λ , we have the types

$$\begin{aligned} \text{(a)} \quad & |wax \ w'ax \ 1|, & \text{(b)} \quad & |wax^2 \ w'a \ 1|, & \text{(c)} \quad & |wa^2x^2 \ w' \ 1|, \\ \text{(d)} \quad & |wa^2x \ w'x \ 1|, & \text{(e)} \quad & |wa^2 \ w'x^2 \ 1|. \end{aligned}$$

By (2), (a) reduces to $|wa \ w'a \ x^2|$ and functions of lower degree.

By (3), (b) " " $|wa \ w'ax \ x|$ " " " " "

By (3), (c) " " $|wa^2 \ w'x \ x|$ " " " " "

By (2), in terms of the element a ,

(d) reduces to $|wx \ w'ax \ a|$ " " " " "

By (3), (e) " " $|wa^2x \ w' \ x|$ " " " " "

All of these results can be reduced by (L), except that of (e) when $w'=1$, thus giving $|wa^2x x 1|$. But, by (3) in terms of the element a , this reduces to $|wx ax a|$ (and functions of lower degree) and (L) reduces this form. Hence:

(M) *The complete system will contain no alternating functions in which more than one element occurs to total degree 2.*

We have remaining, for individual consideration, the types:

a_{001}	a_{010}	a_{100}
a_{002}	a_{020}	a_{200}
a_{003}	a_{030}	a_{300}
a_{011}	a_{101}	a_{110}
a_{012}	a_{102}	a_{120}
a_{021}	a_{201}	a_{210}
a_{111}		
a_{112}	a_{121}	a_{211}

a_{001} and a_{002} can be constructed only in forms which vanish identically. a_{003} has, by (D'), the one type, $|x^2 x 1|$, which is obviously irreducible. a_{011} can appear only as $|x a 1|$ and is obviously irreducible. a_{012} appears either as $|x^2 a 1|$ or $|x a x 1|$. The only possible assumption is

$$|x^2 a 1| \text{ or } |x a x 1| = k(x) |x a 1|.$$

If $a_i = x_i$, we have

$$|x^2 x 1| \text{ or } |x x^2 1| = k(x) |x x 1| = 0,$$

which is untrue. By (2),

$$|x a x 1| = |1 a x^2| - (x) |1 a x| - |x^2 a 1| + (x) |x a 1|.$$

Hence, $|x^2 a 1|$ can be taken as the irreducible type.

Interchanging a and x in this form, $a_{021} = |x a^2 1|$ is seen to be irreducible. a_{111} can appear as:

$$(a) \quad |x \quad \lambda a \quad 1| = \sum_0^3 x_0 (\lambda_1 a_1 - \lambda_2 a_2),$$

$$(b) \quad |\lambda x \quad a \quad 1| = \sum_0^3 \lambda_0 x_0 (a_1 - a_2),$$

$$(c) \quad |ax \quad \lambda \quad 1| = \sum_0^3 a_0 x_0 (\lambda_1 - \lambda_2).$$

The sum of (a) and (b) is

$$\begin{aligned} \sum_0^3 x_0 [a_1 (\lambda_1 + \lambda_0) - a_2 (\lambda_2 + \lambda_0)] &= \sum_0^3 x_0 [a_1 ((\lambda) - \lambda_2) - a_2 ((\lambda) - \lambda_1)] \\ &= (\lambda) \sum_0^3 x_0 (a_1 - a_2) - \sum_0^3 x_0 (\lambda_2 a_1 - \lambda_1 a_2) \\ &= (\lambda) |x a 1| - |x a \lambda|. \end{aligned}$$

Hence

$$|\lambda x a 1| = -|x \lambda a 1| + (\lambda) |x a 1| - |x a \lambda|,$$

the last term of which is reducible by (L).

Interchanging λ and a ,

$$|ax \lambda 1| = -|x \lambda a 1| + (a) |x \lambda 1| - |x \lambda a|.$$

Thus, (b) and (c) are expressible in terms of (a) and functions of lower degree.

Assume (a) :

$$|x \lambda a 1| = k(x) |\lambda a 1| + l(a) |x \lambda 1| + m(\lambda) |x a 1|.$$

Let $\lambda_i = x_i$, $a_i = x_i$, then

$$|x x^2 1| = k(x) |x x 1| + l(x) |x x 1| + m(x) |x x 1| = 0,$$

which is untrue. Hence

$a_{111} = |x \lambda a 1|$ is an irreducible function.

a_{112} can appear as

$$\begin{array}{lll} \text{(a)} & |x^2 \lambda a 1|, & \text{(b)} \quad |\lambda a x x 1|, \quad \text{(c)} \quad |\lambda x^2 a 1| \\ \text{(d)} & |ax^2 \lambda 1|, & \text{(e)} \quad |\lambda x ax 1|. \end{array}$$

By (2), (b) reduces to (a) and functions of lower degree.

By (3), (c) “ “ $|\lambda ax x 1|$ and functions of lower degree.

By (3), (d) “ “ $|a \lambda x x 1|$ “ “ “ “ “

By (2), (e) “ “ $|\lambda a x^2 1|$ “ “ “ “ “

Hence (a) is the one form requiring further examination. Adding this to (c), we obtain finally

$$(N) \quad |x^2 \lambda a 1| = -|\lambda x^2 a 1| + (\lambda) |x^2 a 1| - |x^2 a \lambda|,$$

and the right-hand members are reducible.

Complete System.—We have, therefore, the following complete system of alternating functions:

$$\begin{array}{ll} \text{Of } \lambda \text{ alone} & a_{300} = |\lambda^2 \lambda 1|. \\ \text{Of } a \text{ alone} & a_{030} = |a^2 a 1|. \\ \text{Of } x \text{ alone} & a_{003} = |x^2 x 1|. \\ \text{Of } a \text{ and } x & a_{011} = |x a 1|, \quad a_{012} = |x^2 a 1|, \quad a_{021} = |x a^2 1|. \\ \text{Of } \lambda \text{ and } x & a_{101} = |x \lambda 1|, \quad a_{102} = |x^2 \lambda 1|, \quad a_{201} = |x \lambda^2 1|. \\ \text{Of } \lambda \text{ and } a & a_{110} = |\lambda a 1|, \quad a_{120} = |\lambda a^2 1|, \quad a_{210} = |\lambda^2 a 1|. \\ \text{Of } \lambda, a \text{ and } x & a_{111} = |x \lambda a 1|. \end{array}$$

Furthermore, the foregoing processes enable us to express any given alternating function in terms of the members of the complete systems of symmetric and of alternating functions.

Section (e): Functions of n Rows of Elements.

The operations explained in Sections (c) and (d) are not confined to functions of three rows only. It is easily seen that theorems (A) to (E) inclusive and (A') to (M) inclusive can be re-stated, without material alteration, for a function of n rows of three quantities each.

Consider a symmetric function of four rows, β, λ, a, x . The complete system will contain the irreducible functions of λ, a, x , and similar functions of β, a, x , etc. In addition, we must examine

$$\begin{array}{ll} s_{1111} & \\ s_{1112} & s_{1121}, \text{ etc.}, \\ s_{1122} & s_{1212}, \text{ etc.}, \\ s_{1222}, \text{ etc.}, & \\ s_{2222} & \end{array}$$

$s_{1112} = (\beta_0 \lambda_0 a_0 x_1 x_2)$ reduces by substituting $\beta_i \lambda_i$ for λ_i in (G). Similar methods apply to s_{1122} , s_{1222} , s_{2222} . Assume

$$s_{1111} = (\beta \lambda a x) = k(\beta)(\lambda a x) + m(\lambda)(\beta a x) + \dots$$

Setting $\beta_0 = \beta_1 = \beta_2 = 1$, and collecting terms, we have

$$(\lambda a x) = k'(\lambda a x) + m'(\lambda)(a x) + n'(a)(\lambda x) + p'(x)(\lambda a) + q'(\lambda)(a)(x).$$

If $k' \neq 1$, this gives an expansion for $(\lambda a x)$, which is impossible. If $k' = 1$, we have

$$m'(\lambda)(a x) + n'(a)(\lambda x) + p'(x)(\lambda a) + q'(\lambda)(a)(x) = 0.$$

Let $\lambda_0 = a_0 = 1$, $\lambda_1 = a_1 = -1$, $\lambda_2 = a_2 = 0$, so that $(\lambda) = (a) = 0$. Then

$$2p'(x) = 0, \text{ whence } p' = 0.$$

Similarly, $m' = n' = q' = 0$, so the identity is not true for finite coefficients. Therefore:

Of symmetric functions of finite degree in all four rows, s_{1111} alone is irreducible.

Likewise

$a_{1111} = |\beta \lambda a \ x \ 1|$ is the only irreducible alternating function of finite degree in each of four rows.

Similar statements are readily seen to hold true for 5, 6, ..., n rows.

Section (f): Geometric Classification.

The Three-Point.—The complete system for the three-point consists of those forms, deduced in Sections (c) and (d), which do not contain a . The subscripts denoting degrees in λ and x , respectively, we have

	Symmetric	Alternating	Total
Cubics	s_{03}	a_{03}	2
Conics	$s_{02} \ s_{12}$	a_{12}	3
Lines	$s_{01} \ s_{11} \ s_{21}$	$a_{11} \ a_{21}$	5
Invariants	$s_{10} \ s_{20} \ s_{30}$	a_{30}	4
Total	9	5	14

The Four-Point.—The complete system for the four-point consists of the entire list obtained in Sections (c) and (d). Rearranged according to degree in x , they are:

	Symmetric	Alternating	Total
Cubics	s_{003}	a_{003}	2
Conics	$s_{002} \ s_{012} \ s_{102}$	$a_{012} \ a_{102}$	5
Lines	$s_{001} \ s_{011} \ s_{021} \ s_{201} \ s_{101} \ s_{111}$	$a_{011} \ a_{101} \ a_{111} \ a_{021} \ a_{201}$	11
Invariants	$s_{100} \ s_{010} \ s_{110} \ s_{210} \ s_{120} \ s_{020} \ s_{200}$	$a_{300} \ a_{030} \ a_{120} \ a_{210} \ a_{110}$	14
	$s_{030} \ s_{300}$		
Total	19	13	32

The members of the two systems are the concomitants of the three-point and the four-point under linear transformations sending the line at infinity into itself, either leaving the two points of the absolute fixed, or interchanging them.

Section (g): Syzygies.

It is not our purpose to derive a system of syzygies. However, it appears that the following method is useful in doing so. Take the determinant

$$\begin{vmatrix} (x^2) & x_0^2 & x_1^2 & x_2^2 \\ (\lambda x) & \lambda_0 x_0 & \lambda_1 x_1 & \lambda_2 x_2 \\ (ax) & a_0 x_0 & a_1 x_1 & a_2 x_2 \\ (x) & x_0 & x_1 & x_2 \end{vmatrix} = 0,$$

which vanishes identically because the first column is the sum of the others. Removing the factor $x_0 x_1 x_2$ and expanding:

$$(x^2) |\lambda \ a \ 1| - (\lambda x) |x \ a \ 1| + (ax) |x \ \lambda \ 1| - (x) |x \ \lambda \ a| = 0.$$

Substituting $(x)^2 - 2(x_1x_2)$ for (x^2) , applying (L) to $|x \lambda a|$ and clearing of fractions, we obtain, in the abridged notation,

$$(2s_{001}^2 - 6s_{002})a_{110} + (s_{100}s_{001} - 3s_{101})a_{011} + (3s_{011} - s_{010}s_{001})a_{101} = 0.$$

Section (h): Geometric Interpretation.

In the following we shall denote $\sqrt{\lambda_i}$ by l_i , and $\sqrt{a_i}$ by b_i . Then, it will be remembered, l is the length of a side of the reference triangle, and $b_0, \pm b_1, \pm b_2$, are the coordinates of four points having the reference triangle as their diagonal triangle. So far, we have considered the a 's, λ 's and l 's merely as magnitudes, but, of course, they are also the coordinates of certain points, the positions of which it is well to establish. A point h_0, h_1, h_2 , will be denoted by the symbol h .

It is known that

1 is the centroid of the triangle,

l is the incenter,

λ is the symmedian point—the center of perspective of the vertices of the triangle with the intersections of tangents to the circumcircle at the vertices.

$1/\lambda$: Rays from a vertex through λ and $1/\lambda$ meet the opposite side in points equidistant from its mid-point.

To locate a :

Given points $b_0, \pm b_1, \pm b_2$, the lines joining them, two at a time, are on the vertices of the triangle—two lines on each vertex—and form a harmonic pencil with the two sides of the triangle on the same vertex. Two such lines, on 1, 0, 0, are

$$b_2x_1 - b_1x_2 = 0, \text{ and } b_2x_1 + b_1x_2 = 0,$$

which, taken together, form a degenerate conic

$$b_2^2x_1^2 - b_1^2x_2^2 = a_2x_1^2 - a_1x_2^2 = 0.$$

The polar line of this with respect to 1 is

$$a_2x_1 - a_1x_2 = 0,$$

on which lies a .

From the nature of this construction, the line on 1, 0, 0 and 1, (the median) is the harmonic conjugate of $a_2x_1 - a_1x_2 = 0$ with respect to the two lines of the degenerate conic.

Hence, given a point b , to construct point a :

Join b to a vertex of the reference triangle by a line m and take n , the fourth harmonic of m with respect to the sides of the triangle on that vertex.

Take the fourth harmonic p of the median with respect to m and n . Performing this construction for each vertex, the three lines p will meet in a .

λ^2 : By the above rule λ^2 is obtainable from λ .

It is interesting to note that, if $b_0, \pm b_1, \pm b_2$ are the coordinates of four lines, the line a is on the mid-points of the diagonals of the four-line configuration.

λa : Taking the polar of the degenerate conic $a_2x_1^2 - a_1x_2^2 = 0$, with respect to $1/\lambda$, we have

$$\lambda_2 a_2 x_1 - \lambda_1 a_1 x_2 = 0,$$

on which lies λa . The details of the construction readily follow.

If we calculate the polar systems of $x_0x_1x_2=0$, (the triangle itself) and $|x^2 x 1| = (x_1-x_0)(x_1-x_2)(x_2-x_0)=0$ (the three medians) with respect to the various points defined above, it is seen that they are closely identified with the complete system of Section (f). In several cases, the forms of the complete system and of the polar system are identical, and, most generally, they are members of the same pencil. Hence, the geometrical construction of the complete system can be based upon the construction of the polar system.

The expanded form of $|x^2 x 1|$ in the preceding paragraph shows it to be the discriminant of $x_0x_1x_2=0$. Similarly for $|\lambda^2 \lambda 1|$ and $|a^2 a 1|$.

BIOGRAPHICAL NOTE.

Charles Henry Rawlins, Jr., son of Charles H. and Hester Rawlins, was born in Seaford, Delaware, on the sixteenth day of February, 1889. He acquired his secondary education at the Wilmington Conference Academy, Dover, Del. He was graduated from Dickinson College in 1910, with the degree of Bachelor of Philosophy, and in 1913 he received the degree of Master of Arts from the same institution. During the three years following his graduation he was Instructor in Mathematics at Williamsport (Pa.) Dickinson Seminary. In 1913 he entered The Johns Hopkins University as a graduate student in Mathematics, Astronomy and Physics. During the three years of his course he has held two University Scholarships and the University Fellowship.

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